On Legendre's Function $P_n(\theta)$, *when n is great and θ has any Value.*

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As is well known, an approximate formula for Legendre's function $P_n(\theta)$, when n is very large, was given by Laplace. The subject has been treated with great generality by Hobson,[*] who has developed the complete series proceeding by descending powers of n, not only for P_n but also for the "associated functions." The generality aimed at by Hobson requires the use of advanced mathematical methods. I have thought that a simpler derivation, sufficient for practical purposes and more within the reach of physicists with a smaller mathematical equipment, may be useful. It had, indeed, been worked out independently.

The series, of which Laplace's expression constitutes the first term, is arithmetically useful only when $n\theta$ is at least moderately large. On the other hand, when θ is small, P_n tends to identify itself with the Bessel's function $J_0(n\theta)$, as was first remarked by Mehler. A further development of this approximation is here proposed. Finally, a comparison of the results of the two methods of approximation with the numbers calculated by A. Lodge for $n = 20$[†] is exhibited.

The differential equation satisfied by Legendre's function P_n is

$$\frac{d^2u}{d\theta^2}+\cot\theta\,\frac{du}{d\theta}+n(n+1)\,u = 0. \tag{1}$$

If we assume $u = v(\sin\theta)^{-\frac{1}{2}}$, and write m for $n+\frac{1}{2}$, we have

$$\frac{d^2v}{d\theta^2}+m^2v = -\frac{v}{4\sin^2\theta}. \tag{2}$$

If we take out a further factor, $e^{im\theta}$, writing

$$u = v\sin^{-\frac{1}{2}}\theta = w\,e^{im\theta}\sin^{-\frac{1}{2}}\theta, \tag{3}$$

of which ultimately only the real part is to be retained, we find

$$\frac{d^2w}{d\theta^2}+2im\frac{dw}{d\theta}+\frac{w}{4\sin^2\theta} = 0. \tag{4}$$

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[*] "On a Type of Spherical Harmonics of Unrestricted Degree, Order, and Argument," 'Phil. Trans.,' A, vol. 187 (1896).

[†] "On the Acoustic Shadow of a Sphere," 'Phil. Trans.,' A, vol. 203 (1904); 'Scientific Papers,' vol. 5, p. 163.

We next change the independent variable to z, equal to $\cot\theta$, thus obtaining

$$\frac{dw}{dz} = \frac{-i}{2m}\left\{(1+z^2)\frac{d^2w}{dz^2}+2z\,\frac{dw}{dz}+\frac{w}{4}\right\}. \tag{5}$$

From this equation we can approximate to the desired solution, treating m as a large quantity and supposing that $w = 1$ when $z = 0$, or $\theta = \frac{1}{2}\pi$.

The second approximation gives

$$\frac{dw}{dz} = -\frac{i}{8m}, \qquad \text{whence} \qquad w = 1-\frac{iz}{8m}.$$

After two more steps we find

$$w = 1-iz\left(\frac{1}{8m}-\frac{9}{128m^3}\right)-\frac{9z^2}{128m^2}+\frac{75iz^3}{1024m^3}. \tag{6}$$

Thus in realised form a solution of (1) is

$$u = C\sin^{-\frac{1}{2}}\theta\left[\left(1-\frac{9\cot^2\theta}{128m^2}\right)\cos(m\theta+\gamma)\right.$$
$$\left.+\left\{\frac{\cot\theta}{8m}-\frac{9\cot\theta}{128m^3}-\frac{75\cot^3\theta}{1024m^3}\right\}\sin(m\theta+\gamma)\right]; \tag{7}$$

and this may be identified with P_n provided that the constants C, γ, can be so chosen that u and $du/d\theta$ have the correct values when $\theta = \frac{1}{2}\pi$. For this value of θ we must have

$$P_n(\tfrac{1}{2}\pi) = C\cos(\tfrac{1}{2}m\pi+\gamma), \tag{8}$$

$$(dP_n/d\theta)_{\frac{1}{2}\pi} = C\left(-m-\frac{1}{8m}+\frac{9}{128m^3}\right)\sin(\tfrac{1}{2}m\pi+\gamma). \tag{9}$$

We may express $(dP_n/d\theta)_{\frac{1}{2}\pi}$ by means of $P_{n+1}(\frac{1}{2}\pi)$. In general

$$\sin^2\theta\,\frac{dP_n}{d\cos\theta} = (n+1)(\cos\theta\,P_n-P_{n+1}),$$

so that when $\theta = \frac{1}{2}\pi$,

$$dP_n/d\theta = -dP_n/d\cos\theta = (n+1)P_{n+1}. \tag{10}$$

When n is *even*, $(dP_n/d\theta)_{\frac{1}{2}\pi}$ vanishes, and, C being still undetermined, we may take to satisfy (9), $\gamma = -\frac{1}{4}\pi$; and then from (8)

$$C\sin(\tfrac{1}{2}n\pi) = P_n(\tfrac{1}{2}\pi) = (-1)^{\frac{1}{2}n}\frac{1.3.5\ldots(n-1)}{2.4.6\ldots\ n},$$

so that

$$C = \frac{1.3.5\ldots(n-1)}{2.4.6\ldots\ n}.$$

Here n is even, say $2r$, and it is supposed to be great. Thus

$$C = \frac{1.2.3\ldots(2r-1)\ 2r}{2^2.4^2.6^2\ldots\ldots\ldots(2r)^2} = \frac{(2r)!}{2^{2r}(r!)^2};$$

and when r is great,

$$r! = \sqrt{(2\pi r)} \,.\, r^{r}\, e^{-r+1/12r-1/360r^{3}}.$$

Thus

$$\begin{aligned} C &= \frac{1}{\sqrt{(\pi r)}}\,\text{Exp.}\left(\frac{1}{24r}-\frac{1}{360\,.\,8r^{3}}-\frac{1}{6r}+\frac{2}{360r^{3}}\right)\\ &= \frac{1}{\sqrt{(\pi r)}}\left(1-\frac{1}{8r}+\frac{1}{128r^{2}}+\frac{5}{1024r^{3}}\right)\\ &= \sqrt{\left(\frac{2}{\pi n}\right)}\,.\left(1-\frac{1}{4n}+\frac{1}{32n^{2}}-\frac{5}{128n^{3}}\right). \end{aligned} \tag{11}$$

When n is even and with this value of C,

$$\begin{aligned} P_n = C\sin^{-\frac{1}{2}}\theta\Bigg[\left(1-\frac{9\cot^{2}\theta}{128\,m^{2}}\right)\cos\{(n+\tfrac{1}{2})\,\theta-\tfrac{1}{4}\pi\}\\ +\left(\frac{\cot\theta}{8m}-\frac{9\cot\theta}{128m^{3}}-\frac{75\cot^{3}\theta}{1024m^{3}}\right)\sin\{(n+\tfrac{1}{2})\,\theta-\tfrac{1}{4}\pi\}\,. \end{aligned} \tag{12}$$

When n is odd, the same value of γ, viz. $-\frac{1}{4}\pi$, secures the required evanescence in (8), and we may conjecture that the same value of C will also serve. Laplace* indeed was content to determine γ from the case of n odd and C from the case of n even. I suppose it was this procedure that Todhunter† regarded as unsatisfactory. At any rate there is no difficulty in verifying that (9) is satisfied by the same value of C. From that equation and (10),

$$C = (-1)^{\frac{1}{2}(n+1)}\,\frac{(n+1)\,P_{n+1}(\frac{1}{2}\pi)}{m+1/8m-9/128\,m^{3}},$$

and

$$\begin{aligned} (-1)^{\frac{1}{2}(n+1)}\,P_{n+1}(\tfrac{1}{2}\pi) &= \frac{1\,.\,3\,.\,5\ldots\ n}{2\,.\,4\,.\,6\ldots(n+1)}\\ &= \sqrt{\left\{\frac{2}{(n+1)\,\pi}\right\}}\,.\left\{1-\frac{1}{4\,(n+1)}+\frac{1}{32\,(n+1)^{2}}+\frac{5}{128\,(n+1)^{3}}\right\}. \end{aligned}$$

Here, as throughout, $m = n+\frac{1}{2}$, and when we expand these expressions in descending powers of n we recover (11). Equations (11) and (12) are thus applicable to odd as well as to even values of n.

But whether n be even or odd, (12) fails when θ is so small that $n\theta$ is not moderately large. For this case our original equation (1) takes approximately the form

$$\frac{d^{2}u}{d\theta^{2}}+\frac{1}{\theta}\frac{du}{d\theta}+a^{2}u = 0, \tag{13}$$

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* 'Méc. Cel.,' Supplément au V^e^ volume.

† 'Functions of Laplace, etc.,' p. 71.

where a^2 is written for $n(n+1)$; and of this the solution is

$$u = J_0(a\theta). \tag{14}$$

It is evident that the Bessel's function of the second kind, infinite when $\theta = 0$, does not enter, and that no constant multiplier is required, since u is to be unity when $\theta = 0$. For a second approximation we replace (13) by

$$\frac{d^2u}{d\theta^2}+\frac{1}{\theta}\frac{du}{d\theta}+a^2u = \frac{du}{d\theta}\left(\frac{1}{\theta}-\frac{\cos\theta}{\sin\theta}\right) = \frac{\theta}{3}\frac{du}{d\theta} = \frac{a\theta}{3}J_0'(a\theta);$$

or, if $a\theta = z$,

$$\frac{d^2u}{dz^2}+\frac{1}{z}\frac{du}{dz}+u = \frac{z}{3a^2}J_0'(z). \tag{15}$$

In order to solve (15) we assume as usual

$$u = v\,.\,J_0(z). \tag{16}$$

This substitution gives

$$\frac{d^2v}{dz^2}+\frac{dv}{dz}\left(\frac{2J_0'}{J_0}+\frac{1}{z}\right) = \frac{z}{3a^2}\frac{J_0'}{J_0}, \tag{17}$$

a linear equation of the first order in dv/dz. In this

$$\int\left(\frac{2J_0'}{J_0}+\frac{1}{z}\right)dz = \log(zJ_0^2);$$

so that

$$\frac{dv}{dz} = \frac{A}{zJ_0^2}+\frac{1}{3a^2zJ_0^2}\int z^2J_0J_0'\,dz.$$

Here

$$\int z^2J_0J_0'\,dz = \tfrac{1}{2}z^2J_0^2-\int J_0^2z\,dz = \tfrac{1}{2}z^2J_0^2-\tfrac{1}{2}z^2(J_0^2+J_0'^2) = -\tfrac{1}{2}z^2J_0'^2.$$

Thus

$$\frac{dv}{dz} = \frac{A}{zJ_0^2}-\frac{z}{6a^2}\frac{J_0'^2}{J_0^2}, \tag{18}$$

which has now to be integrated again.

$$\int\frac{J_0'^2z\,dz}{J_0^2} = -\int zJ_0'\,d\left(\frac{1}{J_0}\right) = -\frac{zJ_0'}{J_0}+\int\frac{zJ_0''+J_0'}{J_0}dz$$

$$= -\frac{zJ_0'}{J_0}-\int z\,dz = -\frac{zJ_0'}{J_0}-\frac{z^2}{2},$$

regard being paid to the differential equation satisfied by J_0.

Thus

$$v = B+A\int\frac{dz}{zJ_0^2}+\frac{1}{12a^2}(z^2+2zJ_0'/J_0), \tag{19}$$

and

$$u = BJ_0+AJ_0\int\frac{dz}{zJ_0^2}+\frac{1}{12a^2}(z^2J_0+2zJ_0'). \tag{20}$$

For the present purpose A = 0, B = 1; so that for P_n, identified with u, we get

$$P_n(\theta) = J_0(z) + \frac{1}{12\,a^2}\{z^2 J_0(z) + 2z J_0'(z)\}, \tag{21}$$

in which $z = a\theta, \quad a^2 = n(n+1)$.

The functions J_0, $J_0' = -J_1$, are thoroughly tabulated.*

The Table annexed shows in the second column P_{20} calculated from (21) for values of θ ranging from 0° to 35°. The third column gives the results from (11), (12), beginning with $\theta = 10°$. In the fourth column are the values of P_{20} calculated directly by A. Lodge. It will be seen that for $\theta = 15°$ and 20° the discrepancies are small in the fifth place of decimals. For smaller values of θ, the formula involving the Bessel's functions gives the best results, and for larger values of θ the extended form of Laplace's expression. When θ exceeds about 35° the latter formula gives P_{20} correct to six places. For n greater than 20 the combined use of the two methods would of course allow a still closer approximation.

Table for P_{20}.

θ.	Formula (21).	From (11) and (12).	Calculated by Lodge.
°			
0	1·000000	—	1·000000
5	0·346521	—	0·346521
10	−0·390581	−0·390420	−0·390588
15	−0·052776	−0·052753	−0·052772
20	+0·300174	+0·300191	+0·300203
25	−0·078051	−0·078085	−0·078085
30	−0·216914	−0·216997	−0·216999
35	+0·155472	+0·155635	+0·155636
40	—	+0·127328	+0·127328
45	—	+0·193065	+0·193065

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* See Gray and Mathew's 'Bessel's Functions.'